

# The QCD rotator in the chiral limit

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## Abstract

The low lying spectrum of QCD in the  $\delta$ -regime is calculated here in chiral perturbation theory up to NNLO order. The spectrum has a simple form in terms of the pion decay constant  $F$  and a combination of the low energy constants  $\Lambda_1$  and  $\Lambda_2$ . Since measuring low lying stable masses is among the easiest numerical tasks, the results should help fixing these parameters to good precision.

# 1 Introduction and summary

The low lying spectrum of QCD, in a special environment, is that of a simple quantum mechanical rotator [1]. Although amazing as it is, this spectrum will never be measured in QCD experiments. This physics, however can be studied numerically which will deliver precise predictions on some of the low energy constants in chiral perturbation theory [2, 3, 4]. One of the reasons is that measuring low lying stable masses is among the easiest numerical tasks. Further, the QCD rotator lives in a box of size  $L_s \times L_s \times L_s \times (L_t \rightarrow \infty)$ , which creates an infrared-safe environment<sup>1</sup>. This allows to study the chiral limit first and switching on the symmetry breaking terms later. In two-flavor QCD, in the leading order ( $L$ ) of chiral perturbation theory the  $SU(2) \times SU(2) \sim O(4)$  rotator has an inertia  $\Theta$  proportional to the size of the spatial box:  $\Theta = F^2 L_s^3$ . Here  $F$  is the pion decay constant in the chiral limit. In the next-to-leading order ( $NL$ ) the inertia is corrected  $\Theta = F^2 L_s^3 (1 + \sim 1/F^2 L_s^2)$  [7], where  $1/F^2 L_s^2$  is the small expansion parameter. The logarithms and the additional low energy constants  $l_1$  and  $l_2$  enter only in the  $NNL$  order, which is the highest order we consider in this work.

In the environment discussed above ( $\delta$ -regime [1]) it is natural to divide the degrees of freedom into *fast* and *slow* modes. The fast modes can be treated in perturbation theory, while the slow modes build the slowly moving rotator, whose energy excitations are much smaller than those of the standard Goldstone boson excitations which carry finite momenta.

Let us summarize the final results before going over the details. We quote the result for  $N = 4$ , which corresponds to 2-flavor QCD. Up to  $NNL$  order the rotator spectrum in the chiral limit has the form <sup>2</sup>

$$E_l = \frac{1}{2\Theta} l(l+2), \quad l = 0, 1, 2, \dots, \quad (1)$$

where  $N$  refers to the underlying group  $O(N)$ . The corrections from the perturbative expansion enter in the inertia  $\Theta$  which, up to  $NNL$  order in the chiral limit, reads

$$\begin{aligned} \Theta = F^2 L_s^3 & \left\{ 1 - \frac{2}{F^2 L_s^2} \bar{G}^* \right. \\ & + \frac{1}{(F^2 L_s^2)^2} \left[ 0.088431628 \right. \\ & \left. \left. + d0d0\bar{G}^* \frac{1}{3\pi^2} \left( \frac{1}{4} \ln(\Lambda_1 L_s)^2 + \ln(\Lambda_2 L_s)^2 \right) \right] \right\}. \end{aligned} \quad (2)$$

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<sup>1</sup>It is known since a long time in condensed matter physics also that in a finite box at zero temperature the lowest excitations in the spontaneously broken phase are related to the slow precession of the order parameter described by a rotator ref. [5, 6].

<sup>2</sup>Terms with other Casimir forms are expected to enter beyond  $NNL$  order.

Here  $\Lambda_1, \Lambda_2$  are the standard scales related to the bare low energy constants  $l_1, l_2$  in the chiral Lagrangian. The conventions for  $\Lambda_1, \Lambda_2$  are given in section 7.

We are using dimensional regularization (DR) in this work. The constants  $\bar{G}^*$  and  $d0d0\bar{G}^*$  are related to the constrained Green's function  $\bar{D}^*(x)$  and its second time derivative  $\partial_0\partial_0\bar{D}^*(x)$ , which enter the perturbation theory:

$$\bar{D}^*(0) = \frac{1}{L_s^2} \bar{G}^*, \quad \bar{G}^* = -0.2257849591. \quad (3)$$

$$\partial_0\partial_0\bar{D}^*(0) = \frac{1}{L_s^4} d0d0\bar{G}^*, \quad d0d0\bar{G}^* = -0.8375369106. \quad (4)$$

In the Green's function  $\bar{D}^*(x)$  the non-perturbative slow modes (rotator modes) are missing (notation  $D^*$ ) and the UV singularity is also subtracted (notation  $\bar{D}$ ). The precision of the numerical numbers are estimated to be  $10^{-9}$ , or better<sup>3</sup>. Definitions and properties of the Green's functions we use are summarized in section 5.

Although the final result in eq. (2) is very simple, the underlying chiral perturbation theory is not. For this reason it is a good news that an independent calculation is under way using a completely different technique [8]. Including the symmetry breaking contributions is also in progress [9].

## 2 The chiral action and the rotator in leading order

The low energy limit of QCD with two massless quarks  $m_u = m_d = 0$  is described by an effective non-linear  $O(N = 4)$   $\sigma$ -model. The Lagrangian, up to 1-loop level, has the form  $L_{\text{eff}} = L_{\text{eff}}^{(2)} + L_{\text{eff}}^{(4)}$ , where<sup>4</sup>

$$\begin{aligned} L_{\text{eff}}^{(2)} &= \frac{F^2}{2} \partial_\mu \mathbf{S} \partial_\mu \mathbf{S}, \\ L_{\text{eff}}^{(4)} &= -l_1 (\partial_\mu \mathbf{S} \partial_\mu \mathbf{S}) (\partial_\nu \mathbf{S} \partial_\nu \mathbf{S}) - l_2 (\partial_\mu \mathbf{S} \partial_\nu \mathbf{S}) (\partial_\mu \mathbf{S} \partial_\nu \mathbf{S}). \end{aligned} \quad (5)$$

Here  $F, l_1, l_2$  are the bare low energy constants and the  $N$ -component field has unit length

$$S_a(x), \quad a = 0, 1, \dots, (N-1), \quad \mathbf{S}^2(x) = 1. \quad (6)$$

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<sup>3</sup>Of course, no such precision is needed in this perturbation theory. It might help though when comparing with related works in the future.

<sup>4</sup>We assume that rotation symmetry is respected by the regularization. In dimensional regularization this is the case.

The action in eq. (5) might also be interpreted as the effective low energy prescription of a ferromagnet. We consider an  $L_s \times L_s \times L_s \times L_t$  box with spatial volume  $V_s = L_s^3$ , while the (Euclidean) time extension is taken very large,  $L_t \rightarrow \infty$ . In the  $L_s \rightarrow \infty$  limit the system has a net magnetization and massless Goldstone bosons.

We consider a cylinder geometry ( $\delta$ -regime), where  $L_s$  is finite, but sufficiently large so that the (would be) Goldstone bosons dominate the finite size effects. Due to the microscopic magnetic moments, the  $L_s \times L_s \times L_s$  spatial box has a net magnetization on each time slices. Since  $L_s$  is finite, this net magnetization is moving around as the function of the time  $t$ .

In leading order ( $L$ ), which is a good approximation if the dimensionless expansion parameter  $1/(F^2 L_s^2)$  is small, the microscopic magnets on a time slice are parallel and the rotator action from eq. (5) goes over to

$$A_{\text{rot}} = \frac{F^2 V_s}{2} \int dt \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t). \quad (7)$$

Here  $\mathbf{e}(t)$  is the direction of the total magnetization in the internal  $O(N)$  space at the time  $t$ :

$$\mathbf{e}(t) = [e(t)_0, e(t)_1, \dots, e(t)_{N-1}], \quad \mathbf{e}(t)^2 = 1. \quad (8)$$

Equation (7) describes a quantum mechanical  $O(N)$  rotator, the QCD rotator in leading order [1], with a discrete energy spectrum above the ground state:

$$\mathbf{H} = \frac{\mathbf{L}^2}{2\Theta}, \quad \Theta = F^2 V_s, \quad E_l = \frac{1}{2\Theta} l(l + N - 2), \quad l = 0, 1, 2, \dots \quad (9)$$

In eq (9)  $\mathbf{H}$  is the Hamiltonian,  $\mathbf{L}$  is the  $O(N)$  angular momentum and  $\Theta$  is the inertia of the rotator. Our aim is to determine the corrections up to  $NNL$  order, where the low energy constants  $l_1, l_2$  first enter.

### 3 Separating the slow and fast modes

#### 3.1 New integration variables

If  $L_t \sim L_s$ , it is sufficient to take special care of the freely rotating total magnetization [10, 11, 12, 13]. For  $L_t/L_s$  large, however the magnetization on distant time slices in the cylinder might differ significantly. We have to treat these special slow modes non-perturbatively. These modes are the  $k = (k_0, \mathbf{k} = \mathbf{0})$  modes in Fourier space. In this subsection we deal mainly with the measure, while the action is treated in Sections 3.2 and 3.4.

The steps followed here are similar to that used in [7], but it is simpler and exact in every order. Unlike in [7], where lattice regularization was used, we apply here dimensional regularization.

Insert into the path integral the identity

$$1 = \prod_t \int d\mathbf{m}(t) \prod_{a=0}^{N-1} \delta[m^a(t) - \frac{1}{V_s} \int_{\mathbf{x}} S^a(t, \mathbf{x})] , \quad (10)$$

where  $\mathbf{x}$  is the spatial coordinate,  $x = (t, \mathbf{x})$ , and

$$\mathbf{m}(t) = m(t) \mathbf{e}(t), \quad \mathbf{e}^2 = 1, \quad d\mathbf{m}(t) = m(t)^{N-1} dm(t) d\mathbf{e}(t) . \quad (11)$$

The  $O(N)$  vector of unit length  $\mathbf{e}(t)$  is the direction of the 'magnetization'  $\int_{\mathbf{x}} \mathbf{S}(t, \mathbf{x})$  on the time slice  $t$ . The local 'magnets'  $\mathbf{S}(t, \mathbf{x})$  fluctuate around the slow mode  $\mathbf{e}(t)$ .

Introduce the  $O(N)$  rotation  $\Omega(t)$  as

$$\mathbf{e}(t) = \Omega(t) \mathbf{n}, \quad \mathbf{n} = (1, 0, \dots, 0) . \quad (12)$$

Having an  $O(N)$  matrix  $\Omega$  rather than a vector will be convenient in the manipulations below. We consider  $\Omega$  as a function of  $\mathbf{e}(t)$ . Note, however, that the path integral

$$Z = \prod_x \int d\mathbf{S}(x) \delta(\mathbf{S}^2(x) - 1) \prod_t \int dm(t) m(t)^{N-1} \int d\mathbf{e}(t) \delta^N[m(t)\Omega(t)\mathbf{n} - \frac{1}{V_s} \int_{\mathbf{x}} \mathbf{S}(t, \mathbf{x})] e^{-A_{\text{eff}}(\mathbf{S})} , \quad (13)$$

depends only on the first column of the matrix  $\Omega$ .

Introduce new integration variables  $\mathbf{R}$  in the path integral:

$$\mathbf{S}(t, \mathbf{x}) = \Omega(t) \Sigma(t)^T \mathbf{R}(t, \mathbf{x}) , \quad (14)$$

where  $\Sigma(t) \in O(N)$  and  $\mathbf{R}$  has unit length. The matrix  $\Sigma(t)$  is taken to have a special structure:  $\Sigma(t)_{ij} = \bar{\Sigma}(t)_{ij}$ ,  $\Sigma(t)_{00} = 1$ ,  $\bar{\Sigma}(t)_{0i} = 0$ ,  $\bar{\Sigma}(t)_{i0} = 0$ , where  $\bar{\Sigma}(t)$  is an  $O(N-1)$  matrix. This  $O(N-1)$  matrix  $\bar{\Sigma}(t)$  will be chosen conveniently later.

The partition function reads

$$Z = \prod_x \int d\mathbf{R}(x) \delta(\mathbf{R}^2(x) - 1) \prod_t \int dm(t) m(t)^{N-1} \int d\mathbf{e}(t) \delta^N[m(t)\Omega(t)\mathbf{n} - \Omega(t)\Sigma(t)^T \frac{1}{V_s} \int_{\mathbf{x}} \mathbf{R}(t, \mathbf{x})] e^{-A_{\text{eff}}(\Omega\Sigma^T\mathbf{R})} . \quad (15)$$

Using  $\delta^N(\Omega \mathbf{z}) = \delta^N(\mathbf{z})$  and  $\Sigma \mathbf{n} = \mathbf{n}$  gives

$$Z = \prod_x \int d\mathbf{R}(x) \delta(\mathbf{R}^2(x) - 1) \prod_t \int dm(t) m(t)^{N-1} \int d\mathbf{e}(t) \delta^N \left[ m(t) \mathbf{n} - \frac{1}{V_s} \int_{\mathbf{x}} \mathbf{R}(t, \mathbf{x}) \right] e^{-A_{\text{eff}}(\Omega \Sigma^T \mathbf{R})}. \quad (16)$$

eq. (16) shows that on each time slice the vector  $\mathbf{R}$  fluctuates around the  $(1, 0, \dots, 0)$  internal direction:

$$\mathbf{R}(t, \mathbf{x}) = (1, 0, \dots, 0) + \text{small fluctuations}. \quad (17)$$

The small fluctuations can be treated in perturbation theory:

$$\mathbf{R}(t, \mathbf{x}) = [(1 - \mathbf{\Pi}^2(t, \mathbf{x}))^{\frac{1}{2}}, \mathbf{\Pi}(t, \mathbf{x})], \quad (18)$$

where

$$\mathbf{\Pi}(t, \mathbf{x}) = [\Pi(t, \mathbf{x})_1, \dots, \Pi(t, \mathbf{x})_{N-1}] \quad (19)$$

are the small fluctuations. From eqs. (16), (18) follows

$$\frac{1}{V_s} \int_{\mathbf{x}} \Pi(t, \mathbf{x})_i = 0, \quad i = 1, \dots, N-1. \quad (20)$$

The field  $\mathbf{\Pi}(x)$  is small. This fact allows to integrate out the  $\mathbf{\Pi}$ -fields ('fast modes') in a systematic perturbation theory. We should keep in mind that the slow modes  $\mathbf{e}(t)$  were separated. The  $k = (k_0, \mathbf{k} = \mathbf{0})$  modes in Fourier-space are slow and are not part of the fast  $\mathbf{\Pi}$ -fields.

Two of the integrals in eq. (16) are zero in dimensional regularization. The first one is the well known measure which is created as we replace  $\mathbf{R}$  by the  $\mathbf{\Pi}$  integration variable. The second integral, when put in the exponent, has the form  $(N-1) \int_t \ln[1/V_s \int_{\mathbf{x}} ((1 - \mathbf{\Pi}^2(t, \mathbf{x}))^{1/2})]$ , which is zero also. The relevant part of the partition function  $Z$  will be given in Section 4.

### 3.2 The slow and fast modes in the action $A_{\text{eff}}^{(2)}(\Omega \Sigma^T \mathbf{R})$

Here and in Sec. 3.4 we consider the *action* which is a classical object. In the following steps we found it useful to introduce an infinitesimal parameter  $\epsilon$  which disappears at the end.

In Sec. 3.1 we introduced new variables  $\mathbf{S}(t, \mathbf{x}) = \Omega \Sigma^T \mathbf{R}(t, \mathbf{x})$ . Let us write the time derivatives in the leading action  $A_{\text{eff}}^{(2)}$  in the form

$$\partial_t \mathbf{S}(t, \mathbf{x}) \partial_t \mathbf{S}(t, \mathbf{x}) = \frac{2}{\epsilon^2} [1 - \mathbf{S}(t + \epsilon, \mathbf{x}) \mathbf{S}(t, \mathbf{x})]_{\epsilon \rightarrow 0}. \quad (21)$$

Eqs. (14), (21) imply

$$\partial_\mu \mathbf{S}(t, \mathbf{x}) \partial_\mu \mathbf{S}(t, \mathbf{x}) = \partial_\mu \mathbf{R}(t, \mathbf{x}) \partial_\mu \mathbf{R}(t, \mathbf{x}) - \frac{2}{\epsilon^2} Q(t) \mathbf{R}(t + \epsilon, \mathbf{x}) \mathbf{R}(t, \mathbf{x})_{\epsilon \rightarrow 0}, \quad (22)$$

where the  $N \times N$  matrix  $Q(t)$  is expressed in terms of  $\Omega$  and  $\Sigma$

$$Q(t) = V(t + \epsilon) \Sigma(t + \epsilon)^T - 1. \quad (23)$$

Here we introduced the notation

$$V(t + \epsilon) = \Sigma(t) \Omega(t)^T \Omega(t + \epsilon). \quad (24)$$

The  $O(N)$  matrix  $\Sigma(t)$  is defined in terms of the  $O(N - 1)$  matrix  $\bar{\Sigma}(t)$  (see after eq. (14)). We fix this matrix now:

$$\bar{\Sigma}(t)_{ij} = V(t)_{ij} - \frac{V(t)_{i0} V(t)_{j0}}{1 + V(t)_{00}}. \quad (25)$$

It is easy to show that  $\bar{\Sigma} \in O(N - 1)$ , indeed. Further,  $Q(t)$  can be expressed now in terms of  $V(t + \epsilon)$ :

$$\begin{aligned} Q(t)_{00} &= V(t + \epsilon)_{00} - 1, \quad Q(t)_{0i} = -V(t + \epsilon)_{i0}, \quad Q(t)_{i0} = V(t + \epsilon)_{i0}, \\ Q(t)_{ij} &= -\frac{V(t + \epsilon)_{i0} V(t + \epsilon)_{j,0}}{1 + V(t + \epsilon)_{00}}. \end{aligned} \quad (26)$$

Assume now that  $V$  is known at some  $t_0$ , while  $\Omega(t)$  is known for any  $t^5$ . Having  $V(t_0)$ , we can fix  $\bar{\Sigma}(t_0)$  and  $\Sigma(t_0)$  using eq. (25). Actually, we have the following chain

$$V(t_0) \xrightarrow{\text{eq. (25)}} \Sigma(t_0) \xrightarrow{\text{eq. (24)}} V(t_0 + \epsilon) \xrightarrow{\text{eq. (25)}} \Sigma(t_0 + \epsilon) \xrightarrow{\text{eq. (26)}} Q(t_0). \quad (27)$$

At the end of this chain we have  $Q(t_0)$ . In addition, we have  $V(t_0 + \epsilon)$ , so we start the chain again creating  $Q(t_0 + \epsilon)$  and so on.

We shall see that for the rotator spectrum up to NNL order the following three combinations of the matrix  $Q$  are needed only:  $Q(t)_{00}$ ,  $Q(t)_{i0} Q(t)_{i0}$  and  $Q(t)_{i,i}$  (the repeated index  $i$  is summed). These combinations can be expressed in terms of the slow modes  $\mathbf{e}(t)$  in the  $\epsilon \rightarrow 0$  limit:

$$\begin{aligned} \frac{1}{\epsilon^2} Q(t)_{00} &= -\frac{1}{2} \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t), \quad \frac{1}{\epsilon^2} Q(t)_{i,i} = -\frac{1}{2} \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t), \\ \frac{1}{\epsilon^2} Q(t)_{i0} Q(t)_{i0} &= \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t). \end{aligned} \quad (28)$$

Let us demonstrate eq. (28).

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<sup>5</sup>It will soon turn out that only the first column (i.e. the vector  $\mathbf{e}$ ) is needed.

$$\frac{1}{\epsilon^2} \underline{Q(t)_{00}}$$

Using eqs. (26), (23) and  $\Sigma_{0,a} = \delta_{0,a}$  we obtain

$$\frac{1}{\epsilon^2} Q(t)_{00} = \frac{1}{\epsilon^2} (V(t+\epsilon) - 1)_{00} = \frac{1}{\epsilon^2} (\Omega(t)^T \Omega(t+\epsilon) - 1)_{00}. \quad (29)$$

Expanding in  $\epsilon$  gives

$$\frac{1}{2} [\Omega(t)^T \ddot{\Omega}(t)]_{00} = -\frac{1}{2} [\dot{\Omega}(t)^T \dot{\Omega}(t)]_{00} = -\frac{1}{2} \dot{\epsilon}(t) \dot{\epsilon}(t). \quad (30)$$

Here we used

$$2\dot{\Omega}(t)^T \dot{\Omega}(t) + \ddot{\Omega}(t)^T \Omega(t) + \Omega(t)^T \ddot{\Omega}(t) = 0. \quad (31)$$

Turn to the next case:

$$\frac{1}{\epsilon^2} \underline{Q(t)_{i0} Q(t)_{i0}}$$

Consider first

$$\frac{1}{\epsilon} Q(t)_{a0} = \frac{1}{\epsilon} [\Sigma(t) \Omega(t)^T \Omega(t+\epsilon) - \Sigma(t)]_{a0} = [\Sigma(t) \Omega^T(t)]_{ab} \dot{\epsilon}(t)_b, \quad (32)$$

where we used that  $\Sigma(t)_{a0} = \delta_{a0}$ . It follows then

$$\frac{1}{\epsilon} Q(t)_{i0} \frac{1}{\epsilon} Q(t)_{i0} = \frac{1}{\epsilon} Q(t)_{a0} \frac{1}{\epsilon} Q(t)_{a0} = \dot{\epsilon}(t) \dot{\epsilon}(t). \quad (33)$$

Here we used that  $\frac{1}{\epsilon} Q(t)_{00}$  is  $O(\epsilon)$ .

$$\frac{1}{\epsilon^2} \underline{Q(t)_{ii}}$$

eq. (26) implies

$$\frac{1}{\epsilon^2} Q(t)_{ii} = -\frac{1}{\epsilon^2} \frac{V(t+\epsilon)_{i0} V(t+\epsilon)_{i0}}{1 + V(t+\epsilon)_{00}}. \quad (34)$$

Here  $V(t+\epsilon)_{i0} = Q(t)_{i0}$  and the denominator is  $2 + O(\epsilon^2)$ , we obtain from eq. (33) the result eq. (28).

As discussed before, the fast modes are carried by the  $\mathbf{R} = [(1 - \mathbf{\Pi}^2)^{\frac{1}{2}}, \mathbf{\Pi}]$  field, where the  $k = (k_0, \mathbf{k} = \mathbf{0})$  modes are missing. The first term on the r.h.s. of eq. (22) gives the fast  $\Pi - \Pi$  interactions, while the second term describes the fast-low interactions. We get for the leading action  $A_{\text{eff}}^{(2)}$  in eq. (5):

$$\begin{aligned} A_{\text{eff}}^{(2)}(\Omega \Sigma^T \mathbf{R}) = & \int_x \frac{F^2}{2} \left\{ \partial_\mu \mathbf{R}(x) \partial_\mu \mathbf{R}(x) \right. \\ & - \frac{2}{\epsilon^2} Q(t)_{00} [1 - \mathbf{\Pi}^2(x)] - \frac{2}{\epsilon^2} Q(t)_{ij} \Pi(x)_i \Pi(x)_j \\ & \left. - \frac{2}{\epsilon} Q(t)_{i0} [\partial_t (1 - \mathbf{\Pi}^2(x))^{\frac{1}{2}} \Pi(x)_i - \partial_t \Pi(x)_i (1 - \mathbf{\Pi}^2(x))^{\frac{1}{2}}] \right\}. \end{aligned} \quad (35)$$



The terms above give the action  $A_{\text{eff}}^{(2)}$  up to NNL order. We shall expand in the  $\Pi$ -fields in perturbation theory. The last term in eq. (35), which is odd in  $\Pi$ , enters only in the NNL order of this expansion.

### 3.3 Counting rules

The small expansion parameter in the  $\delta$ -regime is  $1/F^2 L_s^2 = O(\delta^2)$ . The expansion of the rotator action has the form

$$\int_t \frac{F^2}{2} V_s \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t) (1 + \sim \frac{1}{F^2 L_s^2} + \sim \frac{1}{(F^2 L_s^2)^2} \dots) \quad (36)$$

in the leading L, next-to-leading NL, NNL, ... order.

The finite part of the pairings of the fast modes are  $\langle \Pi \Pi \rangle \sim 1/(F^2 L_s^2)$  and  $\langle \partial_\mu \Pi \partial_\nu \Pi \rangle \sim \delta_{\mu,\nu}/(F^2 L_s^4)$ .

In the expansion there are terms also with quadratic and higher powers of the slow mode  $\dot{\mathbf{e}}$ . The following consideration shows that an additional  $\sim \dot{\mathbf{e}}\dot{\mathbf{e}}$  term in the bracket of eq. (36) is  $O(\delta^6)$ , i.e. beyond our NNL calculation.

The argument is as follows. The leading Lagrangian is  $\sim F^2 L_s^3 \dot{\mathbf{e}}\dot{\mathbf{e}}$  which gives for the conjugate momentum  $\mathbf{L} \sim F^2 L_s^3 \dot{\mathbf{e}}$ . It follows then  $\dot{\mathbf{e}}\dot{\mathbf{e}} \sim \mathbf{L}^2 (F^2 L_s^3)^{-2}$ . Since  $\mathbf{L}^2 \sim O(1)$ , we obtain

$$\dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t) L_s^2 \sim \left( \frac{1}{F^2 L_s^2} \right)^2 = O(\delta^4), \quad (37)$$

giving  $\dot{\mathbf{e}}\dot{\mathbf{e}} \sim O(\delta^6)$  as stated above. In the NNNL order, which is beyond our calculation, such corrections are expected to enter. Consider an example. As we shall see later, in the Boltzmann factor enters (among others)

$$\begin{aligned} \exp \left[ \int_x \frac{F^2}{2} \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t) \Pi(x) \Pi(x) \right] &= 1 + \int_x \frac{F^2}{2} \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t) \Pi(x) \Pi(x) \\ &+ \frac{1}{2} \int_x \frac{F^2}{2} \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t) \Pi(x) \Pi(x) \int_{x'} \frac{F^2}{2} \dot{\mathbf{e}}(t') \dot{\mathbf{e}}(t') \Pi(x') \Pi(x') + \dots \end{aligned} \quad (38)$$

We pair out the fast modes on the r.h.s. of eq. (38). Pairing  $\Pi(x)$  with  $\Pi(x)$  and  $\Pi(x')$  with  $\Pi(x')$  in the second line is just needed for the exponentialization. Pairing  $\Pi(x)$  with  $\Pi(x')$  gives, however

$$\begin{aligned} &\sim \int_t \int_{t'} [\dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t)] [\dot{\mathbf{e}}(t') \dot{\mathbf{e}}(t')] \int_{\mathbf{x}} \int_{\mathbf{x}'} D^*(x - x')^2 = \\ &\int_t \int_{t''} F^2 L_s^3 [\dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t)] [\dot{\mathbf{e}}(t - t'') \dot{\mathbf{e}}(t - t'')] \int_{x''} D^*(x'')^2 \frac{1}{F^2}. \end{aligned} \quad (39)$$

In  $D^*(x'')$  the  $\mathbf{k} = 0$  is missing, the smallest  $\mathbf{k}$  is  $\sim 1/L_s$ . Therefore,  $D^*(x'')$  has an exponential cut on the level  $t - t'' \sim L_s$ . The time distance  $L_s$  is small for the rotator, therefore  $\dot{\mathbf{e}}(t - t'') \sim \dot{\mathbf{e}}(t)$ . The integral over  $D^*$ , after renormalization, is  $O(1)$  giving

$$\sim \int_t F^2 L_s^3 [\dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t)] \frac{1}{F^2} [\dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t)] = O(\delta^6). \quad (40)$$

### 3.4 The action $A_{\text{eff}}^{(4)}(\Omega \Sigma^T \mathbf{R})$

In the term with  $l_1$  in eq. (5) we can use the result in eq. (22)

$$\begin{aligned} \partial_\mu \mathbf{S}(x) \partial_\mu \mathbf{S}(x) \partial_\nu \mathbf{S}(x) \partial_\nu \mathbf{S}(x) &= \partial_\mu \mathbf{R}(x) \partial_\mu \mathbf{R}(x) \partial_\nu \mathbf{R}(x) \partial_\nu \mathbf{R}(x) - \\ 2 \partial_\mu \mathbf{R}(x) \partial_\mu \mathbf{R}(x) \frac{2}{\epsilon^2} Q(t) \mathbf{R}(t + \epsilon, \mathbf{x}) \mathbf{R}(t, \mathbf{x}) &+ \\ \frac{2}{\epsilon^2} Q(t) \mathbf{R}(t + \epsilon, \mathbf{x}) \mathbf{R}(t, \mathbf{x}) \frac{2}{\epsilon^2} Q(t) \mathbf{R}(t + \epsilon, \mathbf{x}) \mathbf{R}(t, \mathbf{x}) &\xrightarrow{\epsilon \rightarrow 0}. \end{aligned} \quad (41)$$

The leading term of the first part on the r.h.s. in eq. (41) is  $\partial_\mu \mathbf{\Pi} \partial_\mu \mathbf{\Pi} \partial_\nu \mathbf{\Pi} \partial_\nu \mathbf{\Pi}$ . This term can influence the low mode spectrum only if it is multiplied with both low and fast modes like  $\dot{\mathbf{e}} \dot{\mathbf{e}} \mathbf{\Pi} \mathbf{\Pi}$ . This combination is far beyond our calculation. Up to NNL order we can write for the  $l_1$  part

$$\begin{aligned} \partial_\mu \mathbf{S}(x) \partial_\mu \mathbf{S}(x) \partial_\nu \mathbf{S}(x) \partial_\nu \mathbf{S}(x) &= -\frac{4}{\epsilon^2} Q(t)_{00} \partial_\mu \mathbf{\Pi}(x) \partial_\mu \mathbf{\Pi}(x) \\ &+ \frac{4}{\epsilon^2} Q(t)_{i0} Q(t)_{j0} \partial_0 \mathbf{\Pi}(x)_i \partial_0 \mathbf{\Pi}(x)_j. \end{aligned} \quad (42)$$

The term with  $l_2$  in eq. (5) requests somewhat more, but trivial work. It has the form

$$\begin{aligned} \partial_\mu \mathbf{S}(x) \partial_\nu \mathbf{S}(x) \partial_\mu \mathbf{S}(x) \partial_\nu \mathbf{S}(x) &= -\frac{4}{\epsilon^2} Q(t)_{00} \partial_0 \mathbf{\Pi}(x) \partial_0 \mathbf{\Pi}(x) \\ &+ \frac{4}{\epsilon^2} Q(t)_{i0} Q(t)_{j0} \left\{ \partial_0 \mathbf{\Pi}(x)_i \partial_0 \mathbf{\Pi}(x)_j + \frac{1}{2} \sum_{\alpha=1}^3 \partial_\alpha \mathbf{\Pi}(x)_i \partial_\alpha \mathbf{\Pi}(x)_j \right\}. \end{aligned} \quad (43)$$

The part of the action  $A_{\text{eff}}^{(4)}$  which is relevant up to NNL order has the form

$$\begin{aligned} A_{\text{eff}}^{(4)}(\Omega \Sigma^T \mathbf{R}) &= \\ \int_x \left\{ l_1 \left[ \frac{4}{\epsilon^2} Q(t)_{00} \partial_0 \mathbf{\Pi}(x) \partial_0 \mathbf{\Pi}(x) - \frac{4}{\epsilon^2} Q(t)_{i0} Q(t)_{j0} \partial_0 \mathbf{\Pi}(x)_i \partial_0 \mathbf{\Pi}(x)_j \right] \right. \\ &+ l_2 \left[ -\frac{4}{\epsilon^2} Q(t)_{i0} Q(t)_{j0} \left( \partial_0 \mathbf{\Pi}(x)_i \partial_0 \mathbf{\Pi}(x)_j + \frac{1}{2} \sum_{\alpha=1}^3 \partial_\alpha \mathbf{\Pi}(x)_i \partial_\alpha \mathbf{\Pi}(x)_j \right) \right. \\ &\left. \left. \frac{4}{\epsilon^2} Q(t)_{00} \partial_0 \mathbf{\Pi}(x) \partial_0 \mathbf{\Pi}(x) \right] \right\}_{\epsilon \rightarrow 0}. \end{aligned} \quad (44)$$

## 4 The path integral up to NNL order

The relevant part of the action is the sum of the terms on the r.h.s. of the equations (35) and (44). This action goes in the path integral as  $\exp(-A_{\text{eff}}^{(2)} - A_{\text{eff}}^{(4)})$ , where the action depends on the slow  $\mathbf{e}(t)$  and the fast  $\mathbf{\Pi}(x)$  degrees of freedom. We consider the partition function eq. (16) in dimensional regularization (DR,  $\overline{MS}$ ). Among other conveniences, this simplifies the measure. The Boltzmann factor  $\exp(-A_{\text{eff}}^{(2)} - A_{\text{eff}}^{(4)})$  is expanded in the  $\mathbf{\Pi}$  fields: only the leading parts remain in the exponent, the rest is a Taylor expansion in powers of  $\mathbf{\Pi}$ .

The path integral has the following form up to NNL order<sup>6</sup>:

$$\begin{aligned}
Z = & \prod_t \int d\mathbf{e}(t) \prod_{\mathbf{x}} \int d\mathbf{\Pi}(t, \mathbf{x}) \prod_{i=1}^{N-1} \delta\left(\frac{1}{V_s} \int_{\mathbf{y}} \Pi_i(t, \mathbf{y})\right) \\
& \exp\left[-\int_t \frac{F^2}{2} V_s \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t)\right] \exp\left[-\int_x \frac{F^2}{2} \partial_\mu \mathbf{\Pi}(x) \partial_\mu \mathbf{\Pi}(x)\right] \\
(1) \quad & \left\{ 1 + \int_x \frac{F^2}{2} \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t) \mathbf{\Pi}^2(x) + \dots \right\} \\
(2) \quad & \left\{ 1 + \int_x \frac{F^2}{2} \frac{2}{\epsilon^2} Q(t)_{ij} \Pi(x)_i \Pi(x)_j + \dots \right\} \\
(3) \quad & \left\{ 1 + \frac{1}{2!} \left[ \int_x F^2 \frac{1}{\epsilon} Q(t)_{i0} \left( \frac{1}{2} \mathbf{\Pi}^2(x) \partial_0 \Pi(x)_i - \mathbf{\Pi}(x) \partial_0 \mathbf{\Pi}(x) \Pi(x)_i \right) \right. \right. \\
& \left. \left. \int_y F^2 \frac{1}{\epsilon} Q(y)_{j0} \left( \frac{1}{2} \mathbf{\Pi}^2(y) \partial_0 \Pi(y)_j - \mathbf{\Pi}(y) \partial_0 \mathbf{\Pi}(y) \Pi(y)_j \right) \right] + \dots \right\} \quad (45) \\
(4) \quad & \left\{ 1 - \int_x \frac{F^2}{2} (\mathbf{\Pi}(x) \partial_\mu \mathbf{\Pi}(x)) (\mathbf{\Pi}(x) \partial_\mu \mathbf{\Pi}(x)) + \dots \right\} \\
(5) \quad & \left\{ 1 + 4l_1 \int_x \frac{1}{2} \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t) \partial_\mu \mathbf{\Pi}(x) \partial_\mu \mathbf{\Pi}(x) \right. \\
& \left. + \frac{1}{\epsilon^2} Q(t)_{i0} Q(t)_{j0} \partial_0 \Pi(x)_i \partial_0 \Pi(x)_j + \dots \right\} \\
(6) \quad & \left\{ 1 + 4l_2 \int_x \frac{1}{2} \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t) \partial_0 \mathbf{\Pi}(x) \partial_0 \mathbf{\Pi}(x) + \frac{1}{\epsilon^2} Q(t)_{i0} Q(t)_{j0} \right. \\
& \left. \left( \partial_0 \Pi(x)_i \partial_0 \Pi(x)_j + \frac{1}{2} \sum_{\alpha=1}^3 \partial_\alpha \Pi(x)_i \partial_\alpha \Pi(x)_j \right) + \dots \right\}.
\end{aligned}$$

The integration variables in this path integral are the slow modes  $\mathbf{e}(t)$  and the fast  $\mathbf{\Pi}(x)$  modes with the constraint  $1/V_s \int_{\mathbf{x}} \Pi(t, \mathbf{x})_i = 0$ , i.e. in Fourier

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<sup>6</sup>To make sure the notation: the brackets  $\{ \dots \}$  are multiplied with each other.

space the  $(k_0, \mathbf{k} = 0)$ -modes are missing. The corresponding pairings are

$$< \Pi(x)_i \Pi(0)_j > = \frac{\delta_{ij}}{F^2} D^*(x), < \partial_0 \partial_0 \Pi(x)_i \Pi(0)_j > = \frac{\delta_{ij}}{F^2} \partial_0 \partial_0 D^*(x). \quad (46)$$

In eq. (45) the six lines numbered as (1) ... (6) correspond to the expansion of the action in the exponent. We gave explicitly the terms only which are needed up to NNL. We integrate out in eq. (45) the  $\Pi$ -fields and obtain a path integral in quantum mechanics for a rotator. The first two contributions below are NL<sup>7</sup>, the other five are NNL:

$$\begin{aligned} (1) & \left\{ 1 + \int_t \frac{1}{2} F^2 V_s \dot{\epsilon}(t) \dot{\epsilon}(t) \frac{N-1}{F^2} D^*(0) \right\} \\ (2) & \left\{ 1 - \int_t \frac{1}{2} F^2 V_s \dot{\epsilon}(t) \dot{\epsilon}(t) \frac{1}{F^2} D^*(0) \right\} \\ (1) - (4) \text{ crossing} & \left\{ 1 - \int_t \frac{1}{2} F^2 V_s \dot{\epsilon}(t) \dot{\epsilon}(t) \frac{N-1}{F^4} D^*(0) D^*(0) \right\} \\ (2) - (4) \text{ crossing} & \left\{ 1 + \int_t \frac{1}{2} F^2 V_s \dot{\epsilon}(t) \dot{\epsilon}(t) \frac{1}{F^4} D^*(0) D^*(0) \right\} \\ (3) & \left\{ 1 - \int_t \frac{1}{2} F^2 V_s \dot{\epsilon}(t) \dot{\epsilon}(t) \frac{2N-4}{F^4} \int_z \partial_0 \partial_0 D^*(z) D^*(z) D^*(z) \right\} \\ (5) & \left\{ 1 - 4 l_1 \int_t \frac{1}{2} F^2 V_s \dot{\epsilon}(t) \dot{\epsilon}(t) \frac{2}{F^4} \partial_0 \partial_0 D^*(0) \right\} \\ (6) & \left\{ 1 - 4 l_2 \int_t \frac{1}{2} F^2 V_s \dot{\epsilon}(t) \dot{\epsilon}(t) \frac{N}{F^4} \partial_0 \partial_0 D^*(0) \right\} \end{aligned}$$

Bringing these contributions in the exponent, we get a standard rotator like in eq. (7). Only the inertia  $\Theta$  has corrections:

$$\begin{aligned} \Theta = F^2 V_s & \left\{ 1 - \frac{N-2}{F^2} D^*(0) + \frac{N-2}{F^4} D^*(0) D^*(0) \right. \\ & \left. + 2 \frac{N-2}{F^4} \int_x \partial_0 \partial_0 D^*(x) D^*(x) D^*(x) + \frac{1}{F^4} (8l_1 + 4Nl_2) \partial_0 \partial_0 D^*(0) \right\} \quad (47) \end{aligned}$$

In the following we consider  $N = 4$ , corresponding to two-flavor QCD. Since  $D^*(0)$  and  $\partial_0 \partial_0 D^*(0)$  are known, the only remaining task is to determine the

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<sup>7</sup>The NL result is in agreement with that in [7].

integral in eq. (47). This integral is UV-divergent. The singularity should be canceled by the divergent part of the low energy constants  $l_1$  and  $l_2$ . The singularities of these constants are known in dimensional regularization [4].

## 5 Green's functions

We are in  $d$  dimension,  $d = 4 + \epsilon$ . The physical space-time in Euclidean space is  $L_s \times L_s \times L_s \times (L_t \rightarrow \infty)$  and the corresponding Green's function is  $D(x; d)$ . Subtracting the ultraviolet and infrared (notation: 'bar' and 'star', respectively) divergences we obtain a finite Green's function

$$\bar{D}^*(0) = D(0; d) - (2\pi)^d \int dk^d \frac{1}{k^2 + m^2} - \left(\frac{1}{L_s}\right)^3 (2\pi)^{d-3} \int dk^{d-3} \frac{1}{k^2 + m^2}. \quad (48)$$

We consider the chiral limit  $m \rightarrow 0$ . The UV subtraction (the second term on the r.h.s. above) is zero in dimensional regularization. Therefore,  $\bar{D}^*(0) = D^*(0)$  and  $\partial_0 \partial_0 \bar{D}^*(0) = \partial_0 \partial_0 D^*(0)$ . For the latter, in the framework of DR, the IR subtraction is zero also:  $\partial_0 \partial_0 \bar{D}^*(0) = \partial_0 \partial_0 D(0)$ .

For numerical purposes the following representation is useful<sup>8</sup>

$$\bar{D}^*(x) = \frac{1}{L_s^2} \int_0^\infty dw \frac{1}{4\pi} e^{-\pi w y_0^2} \left\{ \left[ \Pi_{i=1}^3 S(w, y_i) - e^{-\pi w \mathbf{y}^2} \right] - w^{-\frac{3}{2}} \right\}, \quad (49)$$

where  $y = x/L_s$  and

$$S(w, z) = w^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\pi \frac{1}{w} n^2} \cos(2\pi n z), \quad 0 < w < 1, \quad (50)$$

$$S(w, z) = \sum_{n=-\infty}^{\infty} e^{-\pi w (n+z)^2}, \quad 1 < w < \infty. \quad (51)$$

## 6 The integral $\int_\tau dx \partial_0 \partial_0 D^*(x) D^*(x) D^*(x)$

We calculate this integral using dimensional regularization (DR) in the chiral limit. The region of integration is  $L_s \times L_s \times L_s \times (L_t \rightarrow \infty)$ . We write

$$D^*(x) = \Delta(x) + \bar{D}^*(x), \quad (52)$$

where  $\Delta(x)$  is the infinite volume propagator

$$\Delta(x) = \frac{1}{4\pi^2 r^2}, \quad \partial_0 \partial_0 \Delta(x) = \frac{1}{4\pi^2} (4x_0^2 - r^2) \frac{2}{r^6}. \quad (53)$$

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<sup>8</sup>See, for example, in [14, 13].

The Green's functions  $\bar{D}^*(x)$  and  $\partial_0\partial_0\bar{D}^*(x)$  are free of IR and UV singularities.

The integral falls then into six terms. The following four terms (1,2,3,6) are finite in DR :

$$\int_{\tau} dx \left\{ \partial_0\partial_0\bar{D}^*(x)\bar{D}^*(x)\bar{D}^*(x) + \partial_0\partial_0\Delta(x)\bar{D}^*(x)\bar{D}^*(x) + \right. \\ \left. 2\Delta(x)\partial_0\partial_0\bar{D}^*(x)\bar{D}^*(x) + \partial_0\partial_0\Delta(x)\Delta(x)\Delta(x) \right\} , \quad (54)$$

while the terms below (4,5) are UV singular:

$$\int_{\tau} dx \left\{ \Delta(x)\Delta(x)\partial_0\partial_0\bar{D}^*(x) + 2\partial_0\partial_0\Delta(x)\Delta(x)\bar{D}^*(x) \right\} . \quad (55)$$

It is useful to divide the integration region  $\tau$  into a 'cube' and a 'left-right' region, where 'cube' =  $(-L_s/2, L_s/2)^4$  and 'right' =  $(-L_s/2, L_s/2)^3, (L_s, L_t)$  while 'left' =  $(-L_s/2, L_s/2)^3, (-L_s, -L_t)$ . The left-right region is UV-safe<sup>9</sup>.

Consider the integrals in eq. (54). The first and third integrals are finite and can be integrated as they are. For the second integral we write

$$\int_{\tau} dx \partial_0\partial_0\Delta(x)\bar{D}^*(x)\bar{D}^*(x) = \\ \bar{D}^*(0)^2 \int_{\tau} dx \partial_0\partial_0\Delta(x) + \int_{\tau} dx \partial_0\partial_0\Delta(x) \left( \bar{D}^*(x)^2 - \bar{D}^*(0)^2 \right) \quad (56)$$

The first integral on the r.h.s. of eq. (56) is zero in the cube due to 90° rotation symmetry. The rest in eq. (56) can be integrated as it is. The fourth integral in eq. (54) is also zero over the cube. The rest is finite. Let us list the four integral values (1,2,3,6) with  $L_s^{-4}$  suppressed:

$$-1.228902057d-2, \quad -1.730322906d-2, \quad 2.461536207d-2, \quad 4.1588904d-4. \quad (57)$$

The final sum of the four integrals in eq. (54) is

$$- \frac{1}{L_s^4} 0.00456099852. \quad (58)$$

Turn now to the singular integrals in eq. (55). Write the integral (4) as

$$\partial_0\partial_0\bar{D}^*(0) \left\{ \int_R \Delta(x)\Delta(x) - \int_{R \setminus S} \Delta(x)\Delta(x) \right. \\ \left. + \int_{\tau \setminus S} \Delta(x)\Delta(x) \right\} + \int_{\tau} \Delta(x)\Delta(x) \left( \partial_0\partial_0\bar{D}^*(x) - \partial_0\partial_0\bar{D}^*(0) \right), \quad (59)$$

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<sup>9</sup>The procedure applied here is similar to that used in [14, 13].

where  $R \setminus S$  is the full space-time with a sphere cut around  $x = 0$  and similarly for  $\tau \setminus S$ . The radius of the sphere is less than, or equal to  $L_s$ . Only the first integral in eq. (59) is singular with DR, the rest is finite. The result is for (4)

$$-\frac{1}{L_s^4} \left\{ d_0 d_0 \bar{G}^* \frac{1}{8\pi^2} \left[ \frac{1}{d-4} + \ln\left(\frac{1}{L_s}\right) \right] + 0.009856387107 \right\}. \quad (60)$$

In the second integrand (5) in eq. (55) we follow the steps applied above: expand  $\bar{D}^*(x)$  around  $x = 0$  until the rest gives a finite integral. We write

$$\begin{aligned} \bar{D}^*(x) = & \bar{D}^*(0) + \frac{1}{2V_s} |x_0| + \frac{1}{2} \sum_{\mu, \nu} \left( \partial_\mu \partial_\nu \bar{D}^*(0) x_\mu x_\nu \right) \\ & + \left\{ \bar{D}^*(x) - \bar{D}^*(0) - \frac{1}{2V_s} |x_0| - \frac{1}{2} \sum_{\mu, \nu} \left( \partial_\mu \partial_\nu \bar{D}^*(0) x_\mu x_\nu \right) \right\}. \end{aligned} \quad (61)$$

Only the third term on the r.h.s. of eq. (61) is singular in DR. The result is

$$-\frac{1}{L_s^4} \left\{ d_0 d_0 \bar{G}^* \frac{1}{8\pi^2} \frac{2}{3} \left[ \frac{1}{d-4} + \ln\left(\frac{1}{L_s}\right) \right] + 0.015074639535 \right\}. \quad (62)$$

Collecting the results from eqs. (58), (60), (62) we obtain

$$\begin{aligned} \int_\tau dx \partial_0 \partial_0 D^*(x) D^*(x) D^*(x) = \\ -\frac{1}{L_s^4} \left\{ d_0 d_0 \bar{G}^* \frac{1}{8\pi^2} \frac{5}{3} \left[ \frac{1}{d-4} + \ln\left(\frac{1}{L_s}\right) \right] + 0.029492025146 \right\} \end{aligned} \quad (63)$$

## 7 The final result on the inertia $\Theta$

The singular part of the bare low-energy constants  $l_i, i = 1, 2$  are known in dimensional regularization [4]. For the finite part we follow a generally accepted convention [15]. The bare low-energy constants  $l_i$  are written as

$$l_i = \gamma_i \lambda + l_i^r \quad (64)$$

where  $\gamma_1 = \frac{1}{3}$ ,  $\gamma_2 = \frac{2}{3}$  and

$$\lambda = \frac{1}{16\pi^2} \left\{ \frac{1}{d-4} - \frac{1}{2} \left( \ln(4\pi) - C + 1 - \ln(\mu^2) \right) \right\}. \quad (65)$$

Write the renormalized coupling constants as

$$l_i^r = \frac{\gamma_i}{32\pi^2} \ln\left(\frac{\Lambda_i^2}{\mu^2}\right), \quad (66)$$

which gives

$$l_i = \frac{\gamma_i}{16\pi^2} \frac{1}{d-4} - \frac{\gamma_i}{32\pi^2} \left( \ln(4\pi) - C + 1 - \ln(\Lambda_i^2) \right). \quad (67)$$

The combination which enters in eq. (47) is

$$\begin{aligned} \frac{d0d0\bar{G}^*}{(F^2L_s^2)^2} (8l_1 + 16l_2) &= \frac{d0d0\bar{G}^*}{(F^2L_s^2)^2} \left\{ \frac{5}{6\pi^2} \frac{1}{d-4} \right. \\ &\quad \left. - \frac{5}{12\pi^2} \left( \ln(4\pi) - C + 1 \right) + \frac{1}{3\pi^2} \left( \frac{1}{4} \ln(\Lambda_1^2) + \ln(\Lambda_2^2) \right) \right\}. \end{aligned} \quad (68)$$

Multiplying the result in eq. (63) by a factor of 4 and adding to eq. (68) we find that the singularities in eq. (47) cancel and the result in eq. (2) is obtained.

### Acknowledgments

The author is indebted for discussions with G. Colangelo, H. Leutwyler, F. Niedermayer, M. Weingart and Ch. Weyermann. This work is supported in part by the Schweizerischer Nationalfonds. The author acknowledges support by DFG project SFB/TR-55. The "Albert Einstein Center for Fundamental Physics" at Bern University is supported by the "Innovations-und Kooperationsprojekt C-13" of the Schweizerischer Nationalfonds.



## References

- [1] H. Leutwyler, Phys. Lett. **B189**, 197 (1987).
- [2] S. Weinberg, Physica **A96**, 327 (1979).
- [3] J. Gasser and H. Leutwyler, Phys. Lett. **B125**, 321,325 (1983).
- [4] J. Gasser and H. Leutwyler, Ann. Phys. **158**, 142 (1984).
- [5] M. E. Fisher and V. Privman, Phys. Rev. **B32**, 447 (1985).
- [6] E. Brezen and J. Zinn-Justin, Nucl. Phys. **B257**, 867 (1985).
- [7] P. Hasenfratz and F. Niedermayer, Z. Phys. **B92**, 91 (1993).
- [8] F. Niedermayer and Ch. Weiermann, in progress.
- [9] M. Weingart, in progress.
- [10] J. Gasser and H. Leutwyler, Phys. Lett. **B184**, 83 (1987).
- [11] J. Gasser and H. Leutwyler, Phys. Lett. **B188**, 477 (1987).
- [12] J. Gasser and H. Leutwyler, Nucl. Phys. **B307**, 763 (1988).
- [13] P. Hasenfratz and H. Leutwyler, Nucl. Phys. B343, 241 (1990).
- [14] P. Gerber and H. Leutwyler, Nucl. Phys. **B321**, 387 (1989).
- [15] G. Colangelo, J. Gasser and H. Leutwyler, Nucl. Phys. **B603**,125 (2001).